ON PARAMETRIC RELATIONS IN A BALANCED INCOMPLETE BLOCK DESIGN

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1. Introduction

It is well known that besides the two basic parametric relations in a balanced incomplete block design, some other relations must also hold for the existence of a design. Thus Fisher (1940) showed that b must be greater than or equal to v. Bose (1942) showed that for a resolvable balanced incomplete block design $b - r \geqslant v - 1$ and for an affine resolvable design k^2 is divisible by v further. Nair (1943) showed that

$$b - 1 \geqslant \frac{k (r - 1)^2}{r - k + \lambda (k - 1)}.$$

Kishen and Rao (1952) established from complementary designs the the inequalities corresponding to Fisher's and Nair's, *viz.*,

$$b \geqslant v + r - k$$

and

$$b - 1 \geqslant \frac{(v - k)(b - r - 1)^2}{(b - v - r + k) + (b - 2r + \lambda)(v - k - 1)}.$$

It has been shown in this paper that if r and k have no common factor, $b - r \geqslant v - 1$ and the design satisfies the condition of resolvability. In such designs, λ is equal to the remainder left when r is divided by k and $k - 1$ is divisible by λ.

When r and k have a common factor, say, S, as H.C.F. and $r \neq k$, $b - r > v - 1$ if $\lambda < S$; for the equality $b - r = v - 1$, λ must be a multiple of S. $b - r$ can be less than $v - 1$ only when λ is greater than S. Further when S is greater than one, no design can be resolvable. When $S > 1$, a design can be obtained if some value of n_1 can be found so that

$$\frac{r (k - 1)}{r - n_1 . k/S}$$

is an integer.

2. Some Lemmas

In order to get the results it is required to prove the following lemmas:

Lemma 1. $k(b-r)/v-1$ is an integer.

If from a design the r blocks containing any particular treatment are rejected, then the remaining $b-r$ blocks will contain $v-1$ treatments, each treatment being replicated $r-\lambda$ times.

Hence

$$k(b-r)=(v-1)(r-\lambda)$$

i.e., $k(b-r)/v-1$ is an integer being equal to $r-\lambda$.

Lemma 2. $r(b-r)/v-1$ is an integer.

We have

$$\lambda=\frac{r(k-1)}{v-1}$$

$$=\frac{\left(r\,\frac{vr}{b}-1\right)}{v-1}$$

i.e.,

$$b\lambda=\frac{r(vr-b)}{v-1}$$

i.e.,

$$r^2-b\lambda=\frac{r(b-r)}{v-1}$$

which shows that $r(b-r)/v-1$ is an integer.

Lemma 3. $k(v-1)/r$ is an integer.

We have

$$\frac{k(v-1)}{r}=\frac{v(v-1)}{b}=\frac{k(k-1)}{\lambda}$$

i.e.,

$$k(v-1)=\frac{k(k-1)}{\lambda}\cdot r$$

and

$$v(v-1)=\frac{k(k-1)}{\lambda}\,b.$$

Now if $k(k-1)/\lambda$ is not an integer, let λ_1 be the common factor between $k(k-1)$ and λ, so that $k(k-1)=A\lambda_1$ and $\lambda=a\lambda_1$ where A and a are prime to each other. So

$$k(v-1) = \frac{A}{a} r$$

and

$$v(v-1) = \frac{A}{a} b.$$

Hence each of r and b must be divisible by a. So let $r = ar_1$ and $b = ab_1$. The design will, in this case, be of the form

$$v = v,\ b = ab_1,\ k = k,\ r = ar_1 \text{ and } \lambda = a\lambda_1$$

But this is actually the design

$$v = v,\ b = b_1,\ r = r_1,\ k = k \text{ and } \lambda = \lambda_1$$

repeated a times.

Hence in all designs excepting those obtained by repeating a design $k(k-1)/\lambda$, *i.e.*, $k(v-1)/r$ must be an integer.

3 (*a*). WHEN r AND k HAVE NO COMMON FACTOR $b - r \geqslant v - 1$

From lemmas (1) and (2)

$$\frac{b-r}{v-1} = \frac{r-\lambda}{k} = \frac{r^2 - b\lambda}{r} = \frac{n_1}{S} \tag{1}$$

where n_1 is the H.C.F. of $r - \lambda$ and $r^2 - b\lambda$ and S that of r and k, *i.e.*,

$$b - r = \frac{n_1}{S}(v-1).$$

Thus when r and k are prime to each other, $S = 1$ and hence $b - r \geqslant v - 1$.

3 (*b*). λ IS EQUAL TO THE REMAINDER WHEN r IS DIVIDED BY k AND IS ALSO A FACTOR OF $k - 1$

From lemma (3)

$$\frac{v-1}{r} = \frac{N}{k} = \frac{n_2}{S}$$

where N is an integer and n_2 is the H.C.F. of $v - 1$ and N. So when $S = 1$,

$$\frac{v-1}{r} = \frac{k-1}{\lambda} = n_2$$

Hence λ is a factor of $k-1$ and so less than k. Now we have $r = \frac{n_1}{S} k + \lambda$ from (1). Hence when $S = 1$, λ is the remainder left when r is divided by k and n_1 is the quotient.

4. RELATIONS WHEN r AND k HAVE A COMMON FACTOR AND $r \neq k$

We have

$$\frac{n_1}{S} = \frac{r - \lambda}{k}$$

i.e.,

$$n_1 = \frac{r - \lambda}{k_0}$$

where $k_0 = k/S$.

Hence in such designs $r = n_1 k_0 + \lambda$. A design can be obtained in such cases if some value of n_1 can be found such that $r(k-1)/(r - n_1 k_0)$ is an integer.

As

$$\frac{n_1}{S} = \frac{r - \lambda}{k} = \frac{r_0 - \frac{\lambda}{S}}{k_0}$$

where $r = r_0 S$,

n_1/S will be greater than unity if $S > \lambda$. That is for all designs where r and k have a common factor, S, such that $S > \lambda$, $b - r$ will be greater than $v - 1$.

Again n_1/S can be unity, *i.e.*, $b - r$ will be equal to $v - 1$ only when λ is a multiple of S.

Also $b - r$ can be less than $v - 1$ only when $\lambda > S$.

5 (*a*). WHEN r AND k ARE PRIME TO EACH OTHER, THE DESIGN SATISFIES THE CONDITION OF RESOLVABILITY

We have seen that

$$b - r = \frac{n_1}{S}(v - 1)$$

and

$$v - 1 = \frac{n_2}{S} r.$$

So

$$b - r = \frac{n_1 n_2}{S^2}\, r.$$

i.e.,

$$b = \left(1 + \frac{n_1 n_2}{S^2}\right) r.$$

Thus when $S = 1$, the design will have b divisible by r.

5 (*b*). If $S > 1$, the Design cannot be Resolvable

We have seen that

$$b - r = \frac{n_1}{S}(v - 1)$$

and

$$r - \lambda = \frac{n_1}{S}\, k$$

i.e.,

$$\frac{r}{\lambda} = 1 + \frac{n_1}{S} \cdot \frac{k}{\lambda}.$$

Now

$$\frac{b - r}{r} = \frac{n_1}{S} \frac{(v - 1)}{r} = \frac{n_1}{S} \frac{(k - 1)}{\lambda}$$

i.e.,

$$\frac{b}{r} = 1 + \frac{n_1}{S} \frac{k}{\lambda} - \frac{n_1}{S} \cdot \frac{1}{\lambda}$$

$$= \frac{r}{\lambda} - \frac{n_1}{S} \cdot \frac{1}{\lambda}$$

$$= \frac{r - \frac{n_1}{S}}{\lambda}$$

$$= \frac{rk_0 - r_0 + \frac{\lambda}{S}}{\lambda k_0}.$$

Hence b/r cannot be an integer if $S > \lambda$ even though in such cases $b - r > v - 1$.

Again from the equation

$$\frac{b}{r} = \frac{r - \frac{n_1}{S}}{\lambda}$$

We get when $b - r = v - 1$

$$\frac{b}{r} = \frac{r - 1}{\lambda}.$$

As in this case λ is a multiple of S, *i.e.*, say $\lambda = \lambda_0 S$,

$$\frac{b}{r} = \frac{r_0 - \frac{1}{S}}{\lambda_0}.$$

This is evidently a fraction. Thus b/r cannot be an integer when $b - r = v - 1$ and $S > 1$.

Thus it is seen that if r and k have a common factor, no design can be resolvable. The proof, however, does not cover the designs, if any at all, where λ is divisible by S and $r_0 - \lambda/S > k_0$.

6. Summary

If r and k in a B.I.B. design have no common factor $b - r \geqslant v - 1$ and the design satisfies the condition of resolvability. In such designs λ is equal to the remainder left when r is divided by k and $k - 1$ is divisible by λ.

When r and k have common factor, say S, as H.C.F. and $r \neq k$, $b - r > v - 1$ if $\lambda < S$. For the equality $b - r = v - 1$, λ must be a multiple of S. $b - r$ can be less than $v - 1$ only when λ is greater than S. When $S > 1$ a design cannot be resolvable. When $S > 1$ a design can be obtained if some value of n_1 can be found so that

$$\frac{r(k - 1)}{r - n_1 \frac{k}{S}}$$

is an integer.

References

Bose, R. C. .. *Sankhyā*, 1942, **6**, 105–10.

Fisher, R. A. .. *Ann. Eugen. Lond.*, 1940, **10**, 52–75.

Kishen, K. and Rao, C. R. *Jour. Ind. Soc. Agric. Statist.*, 1952, **5**, 137–44.

Nair, K. R. .. *Sankhyā*, 1943, **6**, 255–59.